Diffusion Kernels on q-Gaussian Manifold

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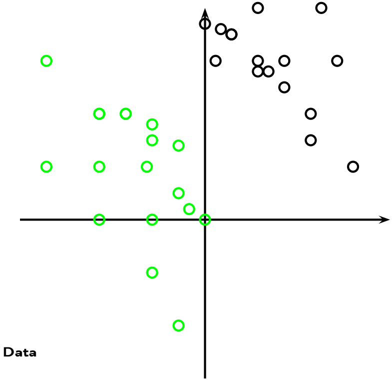
Universidad EAFIT
Department of Mathematical Sciences
PhD in Mathematical Engineering

Objetive of the presentation

A diffusion kernel is a term coined by Laferty (2005) and it alludes to a Mercer kernel (or classifier in the context of Machine Learning), this results from solving the heat equation (diffusion equation) in the modeled manifold in the data set that have a known distribution (multinomial, gaussian, *q*-gaussian, etc.). In this short presentation the path that has been developed to obtain a diffusion kernel will be shown with the hypothesis that the data have a *q*-gaussian distribution with parameters (μ, σ).

Example Support Vector Machine (SVM)

Feature space



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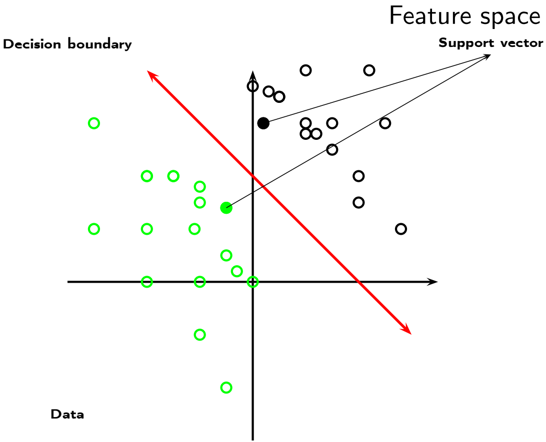

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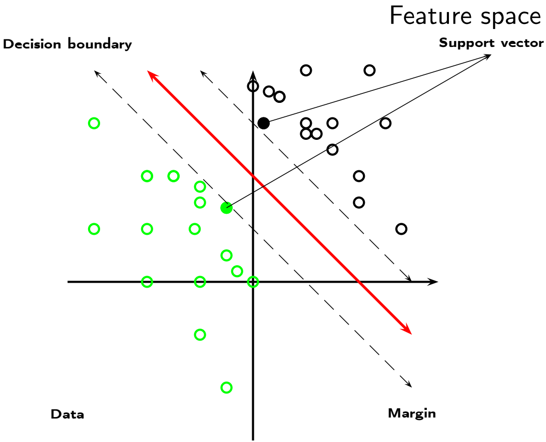

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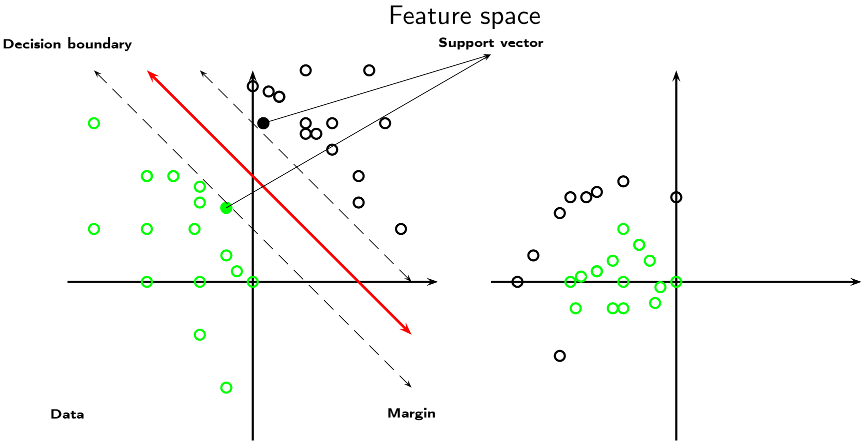

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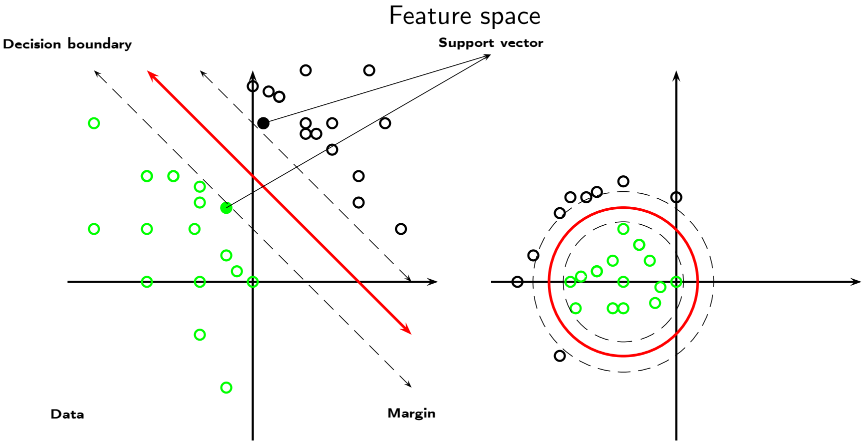

Figure 1: Ideas about the operation of SVM

Concepts, notations and equations of interest

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- $\ell = \log p$: Score function, logarithm of the probability distribution.

Concepts, notations and equations of interest

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$$\Delta_g f = \frac{1}{\sqrt{\det g}} \sum_j \frac{\partial}{\partial x_j} \left(\sum_i g^{ij} \sqrt{\det g} \frac{\partial f}{\partial x_i} \right) \; ,$$

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⋆ $\Gamma_{ij,k}$: Christofell symbols, defined as

$$\Gamma_{ij,k} = \sum_{h=1}^{n} \frac{1}{2} \left[\partial_i g_{jh} + \partial_j g_{ih} - \partial_h g_{ij} \right] g^{hk} \; .$$

- $\star$ R^l_{ijk}: Components of the metric tensor, are calculated by means of the expression

$$R^l_{ijk} = \sum_{h=1}^{n} \left[\Gamma^h_{ik} \Gamma^l_{jh} - \Gamma^h_{jk} \Gamma^l_{ih} \right] + \partial_j \Gamma^l_{ik} - \partial_i \Gamma^l_{jk} \ .$$

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where each θ_i are the components of the parameter θ.

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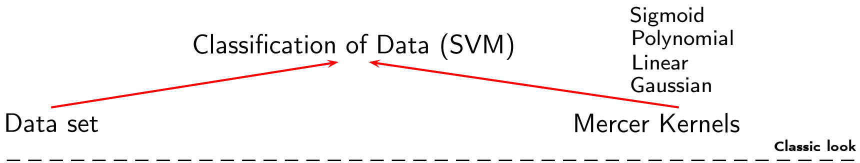
- ⋆ ρ: Geodesic distance, parametrizing the geodetic curve as $\gamma(t)$, this distance is

$$\rho = \int_a^b \sqrt{g_\gamma \left(\dot{\gamma}, \dot{\gamma} \right)} dt$$

where $\dot{\gamma}$ is the derivate of γ with respect to t.

Way to obtain a diffusion kernel

Lafferty and Lebanon propose in their article *Diffusion Kernels on Statistical Manifold* (Laferty and Lebanon, 2005) the following way of work



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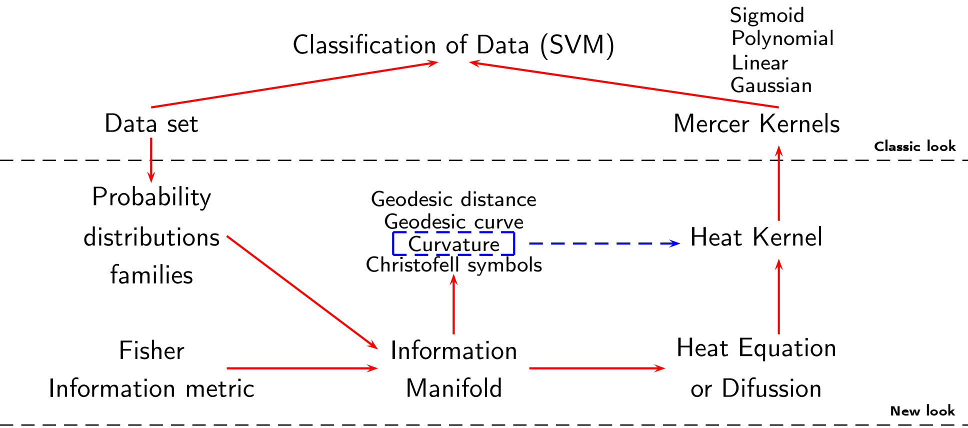

Heat kernel by Grigor'yan and Noguchi

1. In the case of the Euclidean space $\mathbb{R}^n$, the Heat Kernel is given by

$$K_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{\|x-y\|^2}{4t}\right) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{d^2(x,y)}{4t}\right)$$

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2. On the hyperbolic space $\mathbb{H}^n$, the heat kernel is given by

$$K_t(x,x') = \begin{cases} \frac{(-1)^m}{(2\pi)^m} \frac{1}{\sqrt{4\pi t}} \left(\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^m \exp\left(-m^2 t - \frac{\rho^2}{4t}\right) & \text{If } n = 2m+1 \\ \frac{(-1)^m}{(2\pi)^m} \frac{\sqrt{2}}{\sqrt{(4\pi t)^3}} \left(\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^m \int_\rho^\infty \frac{s\exp\left(-\frac{(2m+1)^2 t}{4} - \frac{s^2}{4t}\right)}{\sqrt{\cosh s - \cosh\rho}} ds & \text{If } n = 2m+2 \end{cases}$$

where $\rho = d(x,x')$ is the geodesic distance between the two points in the plane $\mathbb{H}^n$. If $n=2$ ($m=0$ in the second case) then

$$K_t(x,x') = \frac{\sqrt{2}}{(4\pi t)^{\frac{3}{2}}} \exp\left(-\frac{t}{4}\right) \int_\rho^\infty \frac{s\exp\left(-\frac{s^2}{4t}\right)}{\sqrt{\cosh s - \cosh\rho}} ds \, .$$

Tsallis entropy

In the context of non-extensive statistical mechanics, Constantino Tsallis (in 1988) defines entropy relative to q as

$$S_q = \frac{1}{1-q}\left(\sum_i p_i^q - 1\right) = \frac{1}{1-q}\left(h_q - 1\right)$$

where $\sum_i p(x_i) = \sum_i p_i = 1$, q is a fixed value less than 3 called **entropy index** and h_q is the functional $h_q = E[p^q]$ ($E[\cdot]$ it can be summation or integral) that allows defining an expected value relative to q. So, if $q \to 1$ the Shannon entropy

$$S = -\sum_i p\left(x_i\right) \log p\left(x_i\right)$$

used in classical statistical mechanics is obtained.

The description makes sense when defining a pair of inverse functions one of the other, called q-exponential and q-logarithm that generalize the exponential and the logarithm, recovering these when q tends to 1.

The q-exponential function

The q-exponential function is defined as

$$\exp_q(x) = [1 + (1-q)x]_+^{\frac{1}{1-q}}$$

for $-\infty < q < 3$. The derivative for a fixed q value is

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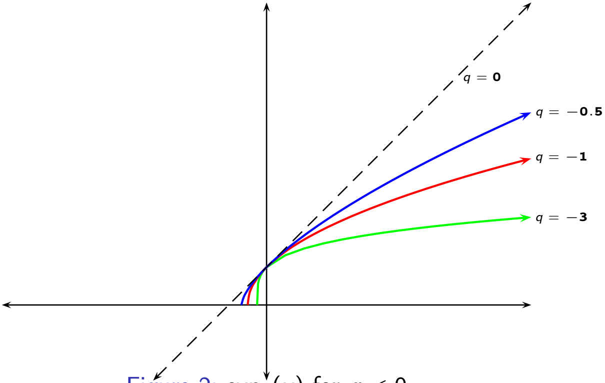

Figure 2: $\exp_q(x)$ for $q < 0$

Graphs for the function q-exponential

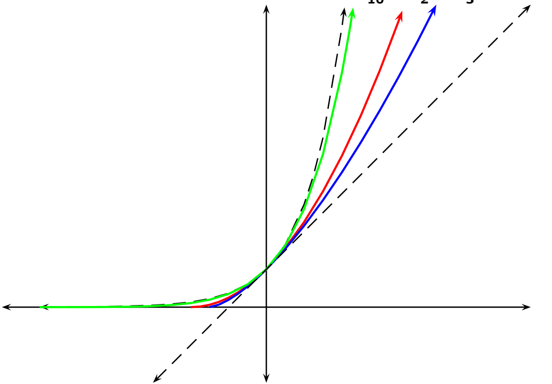

Figure 3: $\exp_q(x)$ for $0 < q < 1$

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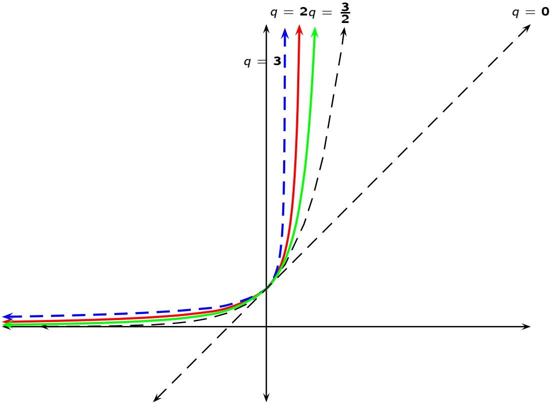

Figure 4: $\exp_q(x)$ for $1 < q < 3$

The q-logarithm function

The inverse of the q-exponential function, the q-logarithm, is given by

$$\ln_q x = \frac{x^{1-q} - 1}{1 - q}$$

provided that $x > 0$. The graph for some values of q is presented below, as well as its derivative for q fixed.

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$\frac{d}{dx}[\log_q(x)] = \frac{1}{x^q}$

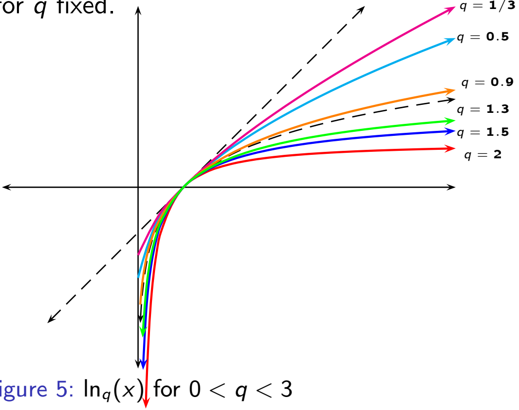

Figure 5: $\ln_q(x)$ for $0 < q < 3$

q-Gaussian Distribution

The q-gaussian distribution has density function

$$p_q(x,\theta) = \frac{1}{Z_{q,\sigma}} \exp_q\left(-\frac{(x-\mu)^2}{(3-q)\sigma^2}\right) = \frac{1}{Z_{q,\sigma}}\left[1 - \frac{(1-q)}{(3-q)}\frac{(x-\mu)^2}{\sigma^2}\right]^{\frac{1}{1-q}}$$

where $\theta = (\mu, \sigma)$ are the parameters on which the manifold of information is defined, $Z_{q,\sigma}$ is the normalization constant that depends on q, is written as $Z_{q,\sigma} = A_q\sigma$.

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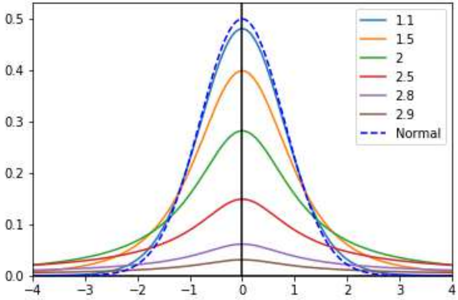

Figure 6: q-gaussian distribution

Normalization constant

The normalization constant is obtained by satisfying the expression

$$\int_{-\infty}^{\infty} f(x)dx = 1 \text{ or } Z_{q,\sigma} = \int_{-\infty}^{\infty} \left[1 - \frac{(1-q)}{(3-q)}\frac{(x-\mu)^2}{\sigma^2}\right]^{\frac{1}{1-q}} dx \ .$$

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By means of a variable change, the constant A_q is defined in terms of the Beta function (or the Gamma function) for some values of q

1. $A_q = \sqrt{\frac{3-q}{1-q}} B\left(\frac{2-q}{1-q}, \frac{1}{2}\right)$ if $-\infty < q < 1$. In this situation the admissible domain for x is $\left[-\frac{\sigma}{\sqrt{1-q}}, \frac{\sigma}{\sqrt{1-q}}\right]$.
2. $A_q = \sqrt{2\pi}$ if $q = 1$.
3. $A_q = \sqrt{\frac{3-q}{q-1}} B\left(\frac{3-q}{2(q-1)}, \frac{1}{2}\right)$ if $1 < q < 3$. The domain for x are all real numbers.

Particular cases

1. Gaussian distribution ($q = 1$).
2. Cauchy distribution ($q = 2$).
3. t-Students distribution ($q = 1 + \frac{2}{n+1}$ with $n \in \mathbb{N}$).
4. Uniform distribution ($q \to -\infty$).
5. Wigner semicircle distribution ($q = -1$).

The q-gaussian distribution belong an exponential family

According to the definition of the function q-logarithm applied to the q-gaussian distribution it is possible to write

$$\log_q p_q = \frac{1}{1-q}\left(\left(\frac{1}{Z_{q,\sigma}}\exp_q\left(-\frac{(x-\mu)^2}{(3-q)\sigma^2}\right)\right)^{1-q} - 1\right) ,$$

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&= \underbrace{\frac{Z_{q,\sigma}^{q-1}}{3-q}\frac{2\mu}{\sigma^2}}_{\theta_1}\underbrace{x}_{F_1(x)} + \underbrace{\frac{Z_{q,\sigma}^{q-1}}{3-q}\frac{1}{\sigma^2}}_{\theta_2}\underbrace{(-x^2)}_{F_2(x)} - \underbrace{\left[\frac{Z_{q,\sigma}^{q-1}}{3-q}\frac{\mu^2}{\sigma^2} - \log_q\left(\frac{1}{Z_{q,\sigma}}\right)\right]}_{\psi_q(\mu,\sigma)} , \\
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Then the q-gaussian distribution is an element in the family q-exponential with parameters and function q-potential

$$\theta_1 = \frac{Z_{q,\sigma}^{q-1}}{3-q}\frac{2\mu}{\sigma^2} , \qquad \theta_2 = -\frac{Z_{q,\sigma}^{q-1}}{3-q}\frac{1}{\sigma^2} ,$$

$$\psi_q(\theta_1,\theta_2) = -\frac{\theta_1^2}{4\theta_2}-\log_q\left[(-d_q\,\theta_2)^{\frac{1}{3-q}}\right] , \qquad \text{with } d_q = \frac{3-q}{A_q^2} .$$

Fisher's metrics and its representations

On the manifold defined by the q-gaussian distribution, it is possible to define two metrics. To do this, Amari (2009) defines the functional Ω_q for a probability distribution p as

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along with the q-expectation

$$E[f(x)] = \int_{\Omega} f(x) \hat{p}_q d\mu = \frac{1}{\Omega_{q,p}} \int_{\Omega} f(x) p^q d\mu \ .$$

For the q-gaussian distribution the relation is fulfilled (Tanaya, 2011)

$$\Omega_{q,p} = \frac{3-q}{2} Z_{q,p}^{1-q} = \frac{3-q}{2} A_q^{1-q} \sigma^{1-q} \ .$$

Fisher's metrics and its representations

One of the metrics defined in the manifold is the usual Fisher g_{ij}^{F} induced by the Score function $\ell = \log p_q$ and the other is the q-Fisher's metric defined by Amari (2009) and what can be written about the distribution $\hat{p}$ as

$$g_{ij}^{(q)} = E_{\hat{p}}\left[(\partial_i \ell_q)(\partial_j \ell_q)\, q p^{q-1}\right] = \frac{q}{\Omega_{q,p}} \int_{\Omega} (\partial_i \ell_q)(\partial_j \ell_q)\, p^{2q-1} d\mu \,.$$

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It is also shown that Fisher's q-metric is of the form

$$g_{ij}^{(q)} = \partial_i \partial_j \psi_q$$

which induces a Hessian manifold. Deriving the function q-potential in terms of parameters θ_1 and θ_2 we get the matrix $g^{(q)}$ and by a change of coordinates it is possible to obtain a matrix diagonal $g_*^{(q)}$.

q-Fisher's metrics and its matrix representations

$$\psi_q(\theta_1,\theta_2) = -\frac{\theta_1^2}{4\theta_2} - \log_q\left[(-d_q\ \theta_2)^{\frac{1}{3-q}}\right]$$

Coordinates (θ_1,θ_2)	Coordinates (μ,σ)
$g^{(q)} = \begin{bmatrix} \frac{-1}{2\theta_2} & \frac{\theta_1}{2\theta_2^2} \\ \frac{\theta_1}{2\theta_2^2} & -\frac{\theta_1^2}{2\theta_2^3} + \frac{1}{3-q}\frac{\Omega_{q,\theta_2}^{-1}}{\theta_2^2} \end{bmatrix}$	$g_*^{(q)} = \begin{bmatrix} \frac{\Omega_{q,\sigma}^{-1}}{\sigma^2} & 0 \\ 0 & \frac{(3-q)\Omega_{q,\sigma}^{-1}}{\sigma^2} \end{bmatrix}$
$\det\left(g^{(q)}\right) = \frac{1}{2(3-q)}\frac{\Omega_{q,\theta_2}^{-1}}{(-\theta_2)^3}$	$\det\left(g_*^{(q)}\right) = \frac{(3-q)\Omega_{q,\sigma}}{\sigma^4}$
$\left[g^{(q)}\right]^{-1} = \begin{bmatrix} (3-q)\Omega\theta_1^2 - 2\theta_2 & (3-q)\Omega\theta_1\theta_2 \\ (3-q)\Omega\theta_1\theta_2 & (3-q)\Omega\theta_2^2 \end{bmatrix}$	$\left[g_*^{(q)}\right]^{-1} = \begin{bmatrix} \Omega\sigma^2 & 0 \\ 0 & \frac{\Omega\sigma^2}{3-q} \end{bmatrix}$

Fisher's metrics and its matrix representations

$$\psi_q(\theta_1,\theta_2) = -\frac{\theta_1^2}{4\theta_2} - \log_q\left[(-d_q\,\theta_2)^{\frac{1}{3-q}}\right]$$

$$g_{ij}^F = \frac{\Omega_{q,p}}{q} g_{ij}^{(q)}$$

Coordinates (θ_1,θ_2)	Coordinates (μ,σ)
$g^F = \begin{bmatrix} \frac{-\Omega}{2q\theta_2} & \frac{\Omega\theta_1}{2q\theta_2^2} \\ \frac{\Omega\theta_1}{2q\theta_2^2} & -\frac{\Omega\theta_1^2}{2q\theta_2^3} + \frac{1}{q(3-q)}\frac{1}{\theta_2^2} \end{bmatrix}$	$g_*^F = \begin{bmatrix} \frac{1}{q\sigma^2} & 0 \\ 0 & \frac{3-q}{q\sigma^2} \end{bmatrix}$
$\det\left(g^{(q)}\right) = \frac{1}{2q^2(3-q)}\frac{\Omega}{(-\theta_2)^3}$	$\det\left(g_*^F\right) = \frac{3-q}{q^2\sigma^4}$
$\left[g^F\right]^{-1} = \begin{bmatrix} (3-q)q\theta_1^2 - \frac{2q\theta_2}{\Omega} & (3-q)q\theta_1\theta_2 \\ (3-q)q\theta_1\theta_2 & (3-q)q\theta_2^2 \end{bmatrix}$	$\left[g_*^F\right]^{-1} = \begin{bmatrix} q\sigma^2 & 0 \\ 0 & \frac{q}{3-q}\sigma^2 \end{bmatrix}$

Christoffel symbols and curvature

Deriving the components of the matrix g_*^F regarding the parameters (μ, σ), it is possible to obtain the Christoffel symbols as summarized in continuation

Derivadas de las componentes de la métrica			
$\partial_1 g_{11}^F = 0$ $\partial_2 g_{11}^F = -\frac{2}{q\sigma^3}$	$\partial_1 g_{12}^F = 0$ $\partial_2 g_{12}^F = 0$	$\partial_1 g_{21}^F = 0$ $\partial_2 g_{21}^F = 0$	$\partial_1 g_{22}^F = 0$ $\partial_2 g_{22}^F = -\frac{2(3-q)}{q\sigma^3}$

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Christoffel symbols			
$\Gamma_{11,1}^F = 0$ $\Gamma_{21,1}^F = -\frac{1}{\sigma}$	$\Gamma_{11,2}^F = \frac{1}{(3-q)\sigma}$ $\Gamma_{21,2}^F = 0$	$\Gamma_{12,1}^F = -\frac{1}{\sigma}$ $\Gamma_{22,1}^F = 0$	$\Gamma_{12,2}^F = 0$ $\Gamma_{22,2}^F = -\frac{1}{\sigma}$

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Metric Tensor		
$R_{212}^1 = \frac{1}{\sigma^2}$	$R_{212}^2 = 0$	$R_{1212} = g_{11} R_{212}^1 = -\frac{1}{q\sigma^4}$

Curvature and Geometry	
$k = \frac{R_{1212}}{\det(g_*^F)} = -\frac{q}{3-q} < 0$	Negative constant curvature (hyperbolic space)

Geodesic curves

Assuming that the coordinates (μ, σ) can be parametric depending on t and with the Christoffel symbols previously found, it is possible to define a system of homogeneous second order differential equations that describes the geodesic curves for the hyperbolic manifold generated by the q-gaussian distribution

$$\frac{d^2\mu}{dt^2} - \frac{2}{\sigma}\frac{d\mu}{dt}\frac{d\sigma}{dt} = 0$$
$$\frac{d^2\sigma}{dt^2} + \frac{1}{(3-q)\sigma}\left(\frac{d\mu}{dt}\right)^2 - \frac{1}{\sigma}\left(\frac{d\sigma}{dt}\right)^2 = 0 \, .$$

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With the substitution $w = \frac{d\mu}{d\sigma}$ it is possible to show that the curve that solves this system is

$$(\mu - h)^2 + (3-q)\sigma^2 = \frac{3-q}{k^2}$$

where h and k are constants that possibly depend on q. This curve is an ellipse with center in $(h, 0)$, that is, on the axis μ. If $q = 2$ (Cauchy distribution) the curves are circumferences of radio $\frac{1}{k}$.

Further works

- ⋆ Find the geodesic distance for a q-gaussiana distribution for any $1 < q < 3$.
- ⋆ The Box-Muller method is applicable for q-gaussian distribution (Thistleton, 2007) generating random data with this distribution.
- ⋆ Program in Python these diffusion kernels for the manifold generated by the q-gaussian distribution.
- ⋆ Define appropriate Christoffel symbols for the q-metric that allow me to find the curvature for the system (μ, σ) and that is in accordance with the result $k = \frac{5-3q}{(q-3)^2(2q-3)}$ (Matsuzoe, 2014).
- ⋆ Study another way to find distances by means of the heat equation (Keenan, 2013).

Bibliographic references

[1] Amari, Shun-ichi and Ohara, Atsumi. *Geometry of q-Exponential Family of Probability Distributions*. Entropy 13, 2011, pp. 1170-1185. doi: 10.3390/e13061170.

[2] Burns; Keith and Gidea; Marian. *Differential Geometry and Topology. With a View to Dynamic System*. Studies in advance mathematics. Chapman & Hall / CRC, 2005.

[3] Cadavid; Carlos and Vélez; Juan Diego. *A Remark on the Heat Equation and Minimal Morse Functions on Tori and Spheres*. Ingeniería y Ciencia, Vol. 09, No. 17, Enero-Junio 2013. EAFIT.

[4] Cardenas Montes; Miguel. *Support Vector Machine. Graphs, Statistics and Data Mining with Python*. Presentation, November of 2015. http://wwwae.ciemat.es/~cardenas/docs/curso_MD/svm.pdf

[5] Grigor'yan, Alexander and Noguchi, Masakazu. *The heat kernel on hyperbolic space*. Bulletin of the London Mathematical Society, 30, pp. 643-650, 1998.

[6] Keenan, Crane. *A New Approach to Computing Distance based on Heat Flow*. NY, 2013.

[7] Laferty; John and Lebanon; Guy. *Difussion Kernels on Statistical Manifolds*. Journal of Machine Learning Research, 6 (2005), pp. 129-163.

[8] Matsuzoe, Hiroshi. *Hessian structures on deformed exponential families and their conformal structures*. Differential Geometry and its Applications, 35 (2014), pp. 323-333.

[9] Matsuzoe, Hiroshi and Ohara, Atsumi. *Geometry for a q-exponential families*. WSPC - Proceedings, April 16, 2011.

[10] *1.4. Support Vector Machines*. Scikit Learn. http://scikit-learn.org/stable/modules/svm.html. September 23/ 2017.

[11] Tanaya, Daiki, Tanaka, Masaru and Matsuzoe, Hiroshi. *Notes on geometry of q-normal distributions*. WSPC - Proceedings, May 07, 2011.

[12] Thistleton, William and others. *Generalized Box-Müller Method for Generating q-Gaussian Random Deviates*. Journal IEEE Transsactions of Information Theory, volume 53, Issue 12, December 2007, pages 4805 - 4810.

Thank you!!